CORRECTION TO *THE WIENER INTEGRAL AND THE SCHRÖDINGER OPERATOR*

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This is a correction of a technical nature to the paper mentioned in the title and hereafter referred to as [0]. Other bibliographic references with the exception of [0] will be the bibliographic references given in [0]. We shall use the notation and results given there through Lemma 2 without further comment.

The error in [0] first appears in the proof of Lemma 3, and affects that part of the proof of Theorem 1 which appears after Lemma 3. Similar difficulties arise in the proof of Theorem 2, but since the proof is essentially the same, we only make a few brief remarks concerning it. The revisions of the theorem that must be made are sufficiently mild that they do not affect the applicability of the results the (formal) Schrödinger differential operators which arise in physics.

Before stating the revised version of Theorem 1 we make the following

DEFINITION. A vector field b on $\boldsymbol{R}^n$ is said to be an *admissible vector potential* if it is of class C^2 and $\int |b|^2 \exp[-\alpha|x|^2]dx < \infty$ for all $\alpha > 0$. An extended real-valued Borel measurable function V on $\boldsymbol{R}^n$ is said to be a *b-amissible scalar potential* if in the notation of [0] we have:

(i) $D(V^+) \cap D(\tilde{A}(b; 0;\ C_0^\infty)$ is dense in $L_2(\boldsymbol{R}^n)$; and

(ii) $D(V^-) \subset C_0^\infty$ and there exists $k > -\infty$ such that for all $\phi \in C_0^\infty(\boldsymbol{R}^n)$ we have

$$(\{-1/2\Delta + V^-\}\phi, \phi) \geqq k. \tag{1}$$

In the above, $V^+ = \operatorname{Max}(V,0)$ and $V^- = \operatorname{Min}(V,0)$.

REMARK. Conditions on V^- which guarantee that (1) holds for some k have been discussed in many places. See [11] for an extensive bibliography.

In the following, the sum of two forms J_1, J_2 on $L_2(\boldsymbol{R}^n)$, for which $D(J_1) \cap D(J_2)$ is dense in $L_2(\boldsymbol{R}^n)$, is defined by: $D(J_1 + J_2) = D(J_1) \cap D(J_2)$ and $(J_1 + J_2)[\phi] = J_1[\phi] + J_2[\phi]$. If A and B are self-adjoint operators in $L_2(\boldsymbol{R}^n)$ such that $D(A) \cap D(B)$ is dense, then $A + B$ is defined in a similar way.

THEOREM 1 (REVISED). *Let b be an admissable vector potential and V a b-admissable scalar potential. Then the form*

$$J_{(b,V)} = J_{\tilde{A}(b;0;C_0^\infty)} + J_{V^+} + J_{V^-}$$

is closable and bounded below by k. If $\hat{A}(b,V)$ is the semi-bounded (by k), self-adjoint operator associated with the closure (Friedrich's extension) of $J_{(b,V)}$, we have that $T_t(ib; V)\phi$ exists a.e. for $\phi \in L_2(\mathbf{R}^n)$ and $t \geqq 0$, and

$$\exp[-\hat{A}(b,V)t]\phi = T_t(ib; V)\phi \quad a.e. \tag{2}$$

Proof. From the proof of Lemmas 1 and 2 in [0] we have that Theorem 1 (revised) is true for V bounded.

LEMMA 3 (REVISED). *Let b and V be as in Theorem 1 (revised). Let V_N, V_N^n be defined as in* [0]. *Then the form*

$$J_{(b,V_N)} = J_{\tilde{A}(b,0,C_0^\infty)} + J_{V_N}$$

is closed and bounded below by k. If $\hat{A}(b,V_N)$ is the semi-bounded (by k) self-adjoint operator associated with $J_{(b,V_N)}$ then $T_t(ib; V)\phi$ exists a.e. for $\phi \in L_2(\mathbf{R}^n)$ and $t \geqq 0$, and

$$\exp[-\hat{A}(b,V_N)t]\phi = T_t(ib; V_N)\phi \quad a.e. \tag{3}$$

Proof of the lemma. Since both $J_{\hat{A}(b;0;C_0^\infty)}$ and J_{V_N} are bounded below and closed, we have, by Theorem 3.1 in [15], that $J_{(b,V_N)}$ is closed. A direct calculation shows that

$$J_{(b;V_N)} = \sup_n J_{\tilde{A}(b;V_N^n C_0^\infty)}.$$

Since $J_{\tilde{A}(b;V_N^n;C_0^\infty)} \uparrow n$ and $J_{\tilde{A}(b;V_N^n;C_0^\infty)} \leqq$ the closable form defined by the semi-bounded symmetric operator $\tilde{A}(b;0;C_0^\infty) + V_N$, we have that the sequence $\{J_{\tilde{A}(b;V_N^n;C_0^\infty)}\}$ is bounded above by a closed form. Thus Theorem 10.1 in [15] is applicable and we have that

$$\lim_{n\to\infty} R_\lambda(-\{\tilde{A}(b;V_N^n;C_0^\infty)\} = R_\lambda(-\hat{A}(b,V_N)) \tag{4}$$

in the strong operator topology for $\mathrm{Re}(\lambda) > N$. The proof in [0] of the fact that $T_t(ib; V_N^n)$ exists a.e. for $\phi \in L_2(\mathbf{R}^n)$, $t \geqq 0$ and

$$\lim_{n\to\infty} \| T_t(ib; V_N^n)\phi - T_t(ib; V_N)\phi \|_2 = 0, \tag{5}$$

is still valid.

(4) and (5) combined with the fact that $\| T_t(ib; V_N^n)\| \leqq e^{Nt}$, imply that

$$R_\lambda(-\hat{A}(b,V_N)) = \int_0^\infty e^{-\lambda t}\, T_t(ib; V_N)\phi\, dt \quad \text{for} \quad \mathrm{Re}(\lambda) > N.$$

This implies by Corollary VIII.1.16 in [3] that (3) is true.

It remains to show that $\hat{A}(b,V_N)$ is bounded below by k. We prove this first for the case $b = 0$. It is clear that $A(0; V_N^n; C_0^\infty) \geqq A(0; V^- C_0^\infty)$ and thus

$$J_{\tilde{A}(0;V_N^n;C_0^\infty)} \geqq J_{\tilde{A}(0;V-C_0^\infty)} \geqq k$$

for all n and N. Thus

$$\sup_n J_{\tilde{A}(0;V_N^n;C_0^\infty)} = J_{(0;V_N)} \geqq k$$

which, of course, implies $\hat{A}(0, V_N) \geqq k$. To prove this for $b \neq 0$, we note that

$$(6) \qquad |T_t(ib;\, V_N)\phi(x)| \leqq T_t(0;\, V_N)|\phi|(x)$$

since, for Wiener integrable functionals f, we have $|E_x(f)| \leqq E_x(|f|)$. (6) implies that $\| T_t(ib;\, V_N)\phi \|_2 \leqq \| T_t(0;\, V_N)|\phi| \|_2 \leqq \| T_t(0;\, V_N)\|_{0p} \| \phi \|_2 \leqq \exp[-kt\|\phi\|_2]$. The last inequality follows from the fact that $-\hat{A}(0;\, V_N) \leqq -k$ and the spectral theorem. Thus

$$(7) \qquad \| T_t(ib;\, V_N) \|_{0p} \leqq e^{-kt}.$$

From the spectral theorem we see that $-\hat{A}(b, V_N) \leqq -k$ i.e. $\hat{A}(b, V_N) \geqq k$ which completes the proof of the lemma.

We now return to the proof of the theorem. The proof in [0] of the fact that $T_t(ib;\, V_N)\phi$ exists a.e. for $\phi \in L_2(\boldsymbol{R}^n)$, $t \geqq 0$.

$$(8) \qquad \lim_{n\to\infty} \| T_t(ib;\, V_N)\phi - T_t(ib;\, V)\phi \|_2 = 0$$

is still valid modulo the truth of (6.11) in [0]. But (7) shows that 6.11 is true, for our case. Since $J_{(b,V_N)} \downarrow_N$ and $J_{(b,V_N)} \geqq k$, we can apply Theorem 10.2 in [15] and thus we have $\inf_N J_{(b,V_N)} \geqq k$ is closable. A direct calculation shows that $\inf_N J_{(b,V_N)} = J_{(b,V)}$. Clearly then $\tilde{J}_{(b,V)} \geqq k$. The theorem also tells us that

$$(9) \qquad \lim_{n\to\infty} R_\lambda(-\hat{A}(b, V_N)) = R_\lambda(-\hat{A}(b, V))$$

in the strong operator topology for $\mathrm{Re}(\lambda) > -k$. Thus (7), (8) and (9) imply that

$$R_\lambda(-\hat{A}(b, V)) = \int_0^\infty e^{-\lambda t} T_t(ib;\, V)\phi \, dt,$$

for $\mathrm{Re}(\lambda) > k$. As has already been noted, this implies (2) and the theorem is proved.

Remarks concerning Theorem 2. If G is an open subset of $\boldsymbol{R}^n$ such that ∂G has Lebesgue measure zero and J is a closed semi-bounded form on $L_2(\boldsymbol{R}^n)$ such that $D(J) \cap L_2(G)$ is dense in $L_2(G)$, then $J_G \equiv J_{|L^2(G)}$ is a closed semi-bounded form on $L_2(G)$. If A is the self-adjoint operator on $L_2(\boldsymbol{R}^n)$ which is associated with J, then A_G will denote the self-adjoint operator on $L_2(G)$ which is associated with $J_{(G)}$.

Definition. A vector field is said to be an *admissible G-vector potential* if it can be extended to an admissible vector potential on $\boldsymbol{R}^n$. An extended real-valued Borel measurable function V on G is a *$b - G$ admissible scalar potential* if

(i) $D(V^+) \cap D((\tilde{A}(b; 0; C_0^\infty))_G)$ is dense in $L_2(G)$ and

(ii) there exists a core $C = C(\tilde{A}(0; 0; C_0^\infty))_G$ of $(\tilde{A}(0; 0; C_0^\infty))_G$ (i.e. a set on which $(\tilde{A}(0; 0; C_0^\infty))_G$ is essentially self-adjoint) and a real number $k(> -\infty)$ such that $D(V^-) \supset C$ and for all $\phi \in C$,

$$(\{(\tilde{A}(0; 0; C_0^\infty))_G + V^-\}\phi, \phi)_G \geqq k.$$

Theorem 2 now reads as follows:

THEOREM 2 (REVISED). *Let G be an open subset of $\boldsymbol{R}^n$ as above. Let b be an admissible G-vector potential and V a $b - G$ admissible scalar potential. Then the form*

$$J_{(b,V;G)} \equiv (J_{\tilde{A}(b;0;C_0^\infty)})_G + J_{V^+} + J_{V^-}$$

is closable and bounded below by k. If $\hat{A}(b, V; G)$ is the semi-bounded (by k) self-adjoint operator associated with $\bar{J}_{(b,V;G)}$, then $T_t(ib; V; G)\phi$ exists a.e. for $\phi \in L_2(\boldsymbol{R}^n)$, $t \geqq 0$ and

$$\exp[-\hat{A}(b, V; G)t]\phi = T(ib; V; G)\phi \quad a.e. \tag{10}$$

As for the proof, once the formula has been established for bounded V, the proof follows exactly as the proof of Theorem 1 (revised). The proof of (10) for bounded V, follows the proof in [0] except $\tilde{A}_G$ is replaced by $\hat{A}(b, V; G)$.

BIBLIOGRAPHY

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